Pufendorf Lectures 2025: **Vocabularies of Reason**

Handout for Lecture III:

Roles and Reasons

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A *vocabulary* is a pair $\langle L,\mathbf{I}\rangle$ of a *lexicon* L, which is a set of sentences, and distinguished set **I** of pairs of sets of sentences of L. $\langle X,Y\rangle\in\mathbf{I}$ means that the implication with premises X and conclusion Y is a good one. Premise-sets are read *con*junctively, and conclusion-sets are read *dis*junctively. We mark the *incoherence* of the premise-set X by $\langle X,\varnothing\rangle\in\mathbf{I}$.

The *implication space* defined by a vocabulary with lexicon L is the set $\mathcal{P}(L)\mathrm{x}\mathcal{P}(L)$ of pairs of sets of sentences of L.
The points of the implication space are thought of as *candidate implications.* The good implications, according to the implication-space frame defined from a vocabulary, are just those in **I**.

"Kant was on the right track when he insisted that just as concepts are essentially (and not accidentally) items which can occur in judgments, so judgments (and, therefore, indirectly concepts) are essentially (and not accidentally) items which can occur in reasonings or arguments."
["Inference and Meaning" [I-4], in Kevin Scharp and Robert Brandom (eds.) *In the Space of Reasons*: *Selected Essays of Wilfrid Sellars* [Harvard University Press, 2007].

- The *extension* of a candidate implication $\langle X,Y\rangle$ is just its *goodness* value: whether or not it is a good implication (to be found in **I**).
- The *intension* or semantic interpretant of a candidate implication is its *range of subjunctive robustness* (RSR): the set of additions to its premise- and conclusion-set that *complete* it, in the sense that they would *make* it good if it is *not* good, and *keep* it good, if it *is* good.

  $$\mathrm{RSR}\langle\Gamma,\Delta\rangle =_{\mathrm{df.}} \{\langle X,Y\rangle\in L\mathrm{x}L : \langle\Gamma\cup X,\Delta\cup Y\rangle\in\mathbf{I}_M.$$

- The *implicational role* of implication $\Gamma|\!\sim\Delta$ is the equivalence class of (sets of) candidate implications that share its range of subjunctive robustness (intension):

  $$\mathcal{R}(\{\langle\Gamma,\Delta\rangle\}) =\mathrm{df.}\ \{\langle X,Y\rangle\in L\mathrm{x}L : \mathrm{RSR}\langle X,Y\rangle = \mathrm{RSR}\langle\Gamma,\Delta\rangle\}.$$

- The *conceptual* (propositional) *content* of sentence $A\in L$ is the pair of the implicational roles of its *premissory* and *conclusory* seed implications $\langle A,\varnothing\rangle$ and $\langle\varnothing,A\rangle$:

  $$[A] = \langle a^+,a^-\rangle = \langle\, \mathcal{R}(\{\langle A,\varnothing\rangle\}), \mathcal{R}(\{\langle\varnothing,A\rangle\})\,\rangle.$$

These are the inferential *consequences* and *circumstances* of application of the sentence A.

Operations on Implicational Roles:

Symjunction: $\mathcal{R}(X) \sqcap \mathcal{R}(Y) \;=_{df.}\; \mathcal{R}(X \cup Y)$.

Adjunction: $\mathcal{R}(X) \sqcup \mathcal{R}(Y) \;=_{df.}\; \mathcal{R}(\{\Gamma \cup \Delta : \Gamma \in X, \Delta \in Y\})$.

The implication-space semantic definitions of the connectives of NMMS:

$[A] =_{df.} \langle a^{+}, a^{-} \rangle$ $\qquad$ $[B] =_{df.} \langle b^{+}, b^{-} \rangle$

$\sqcup$ is adjunction of implicational roles, $\qquad$ $\sqcap$ is symjunction of implicational roles

Computing the conceptual roles of logically complex sentences:

$[\neg A] =_{df.} \langle a^{-}, a^{+} \rangle$.

$[A \rightarrow B] =_{df.} \langle a^{-} \sqcap b^{+} \sqcap (a^{-} \sqcup b^{+}),\; a^{+} \sqcup b^{-} \rangle$.

$[A \& B] =_{df.} \langle a^{+} \sqcup b^{+},\; a^{-} \sqcap b^{-} \sqcap (a^{-} \sqcup b^{-}) \rangle$.

$[A \vee B] =_{df.} \langle a^{+} \sqcap b^{+} \sqcap (a^{+} \sqcup b^{+}),\; a^{-} \sqcup b^{-} \rangle$.

Fact: These implication-space connective definitions in terms of symjunction and adjunction of roles produce a *sound and complete semantics* for the universally LX logic NMMS—the substructurally forgiving version of classical logic.

The Metalogical Correspondence between Implication-Space and Sequent-Calculus MVs:

1. The *first* element in the roles defined by the semantic clauses corresponds to the *left* rule in the sequent calculus, and the *second* element corresponds to the *right* rule in the sequent calculus.
2. The roles super-scripted with a "+" stem from sentences that occur on the *left* in a top sequent, and the roles super-scripted with a "−" stem from sentences that occur on the *right* in a top sequent.
3. An *adjunction* ⊔ indicates that the adjoined roles stem from sentences in a *single* top sequent. And a *symjunction* ⊓ indicates that the symjoined roles stem from sentences that occur in *different* top sequents.

Given that the contexts Γ,Δ are always shared in all the sequents of any rule application, using this correspondence, the implication-space semantic clauses above uniquely determine the sequent rules of any logic, and the other way around.

Connective Rules of NMMS:

L¬: $$\frac{\Gamma \mid\!\sim \Delta, A}{\Gamma, \neg A \mid\!\sim \Delta}$$

R¬: $$\frac{\Gamma, A \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta, \neg A}$$

L→: $$\frac{\Gamma \mid\!\sim \Delta, A \quad B, \Gamma \mid \Delta \quad B, \Gamma \mid\!\sim \Delta, A}{\Gamma, A \rightarrow B \mid\!\sim \Delta}$$

R→: $$\frac{\Gamma, A \mid\!\sim B, \Delta}{\Gamma \mid\!\sim A \rightarrow B, \Delta}$$

L&: $$\frac{\Gamma, A, B \mid\!\sim \Delta}{\Gamma, A \& B \mid\!\sim \Delta}$$

R&: $$\frac{\Gamma \mid\!\sim \Delta, A \quad \Gamma \mid\!\sim \Delta, B \quad \Gamma \mid\!\sim \Delta, A, B}{\Gamma \mid\!\sim \Delta, A \& B}$$

L∨:	$\frac{\Gamma,A\mid\sim\Delta \quad \Gamma,B\mid\sim\Delta \quad \Gamma,A,B\mid\sim\Delta}{\Gamma,A\vee B\mid\sim\Delta}$	R∨:	$\frac{\Gamma\mid\sim\Delta,A,B}{\Gamma\mid\sim\Delta,A\vee B}$

Implicational Role Inclusions:

Premissory Role Inclusion: If $RSR(a^+)\subseteq RSR(b^+)$ then for all contexts
$\Gamma,\Delta\subseteq L[\Gamma,A\mid\sim\Delta \Rightarrow \Gamma,B\mid\sim\Delta]$, so A can be substituted for B as a premise *salva consequentia*.
Conclusory Role Inclusion: : If $RSR(b^-)\subseteq RSR(a^-)$ then for all contexts
$\Gamma,\Delta\subseteq L[\Gamma\mid\sim A,\Delta \Rightarrow \Gamma\mid\sim B,\Delta]$, so A can be substituted for B as a conclusion *salva consequentia*.
If Cut (CT) holds, then $RSR(a^+)\subseteq RSR(b^+)$ iff $RSR(b^-)\subseteq RSR(a^-)$.
'Pedro is a donkey' implies 'Pedro is a mammal,' A|~B, and
'Pedro is a donkey' can be substituted everywhere as a premise for 'Pedro is a mammal', *salva consequentia*, $RSR(a^+)\subseteq RSR(b^+)$, and
'Pedro is a mammal' can be substituted everywhere for 'Pedro is a donkey as a conclusion, *salva consequential*, $RSR(b^-)\subseteq RSR(a^-)$.
Absent that global transitivity structure, premissory and conclusory roles can diverge.

Trilogics:

The paracomplete logic K3 and the paraconsistent logic LP (Graham Priest's 'Logic of Paradox') both use the three-valued Strong Kleene connective definitions, differing only in how they define consequence.
K3 treats an implication as good iff it preserves the value 1 (true), and
LP treats it as good iff it preserves non-0 (non-false) values.
K3 invalidates Excluded Middle, and is a logic of truth-value *gaps*;
Its middle value ½ may be thought of as meaning '*neither* true *nor* false.'
LP invalidates Noncontradiction, and is a logic of truth-value *gluts*;
Its middle value ½ may be thought of as meaning '*both* true *and* false.'
It has been suggested that semantic paradoxes can be dealt with by assigning paradoxical sentences like the Liar the third truth-value of ½ and drawing consequences according to K3 (Kripke) or LP (Priest).

Fact: **K3 is the logic of *premissory* implicational role inclusions and**
LP is the logic of *conclusory* implicational role inclusions.

What shows up in the (extended) semantics of truth-values as the difference between countenancing truth-value *gaps* and *gluts* shows up in the implication-space setting as the difference between *premissory* and *conclusory* role inclusions.

Four Kinds of Rational Metavocabulary:

Extrinsic-Explanatory:

1. Bilateral Deontic Normative Pragmatic Metavocabulary.
   Key concepts: Doxastic commitments to accept/reject, expressed in speech acts of assertion/denial (generically: claimings), entitlement to which commitments can be challenged by giving reasons *against* them (governed by incompatibilities) and defended by giving reasons *for* them (governed by implications). $\Gamma|\sim\Delta$ means one cannot be entitled to commitments to accept all of Γ and reject all of Δ.

2. Truthmaker Alethic Modal Mereological Semantic Metavocabulary.
   Key concepts: Possible/impossible states and their mereological fusions, propositions as pairs of truthmakers and falsitymakers related by Exclusion (every fusion of a truthmaker and a falsitymaker of the same proposition is an impossible state). $\Gamma|\sim\Delta$ means every fusion of a truthmaker of all of Γ with a falsitymaker of all of Δ is an impossible state.

Intrinsic-Explicative:

3. Sequent-Calculus Metavocabulary for NonMonotonic MultiSuccedent (NMMS) Logic.
   Key Concepts: Metainferential rules relating reason relations. Universal LX-ness of NMMS. It is expressively complete in that for every set of implications in any (CO-compliant) base vocabulary there is a sequent in its logical NMMS extension that is good in all and only the models elaborated from base vocabularies in which those atomic sequents hold, and *vice versa*.

4. Implication-Space Semantic Metavocabulary of Conceptual Roles.
   Key Concepts: Ranges of subjunctive robustness, implicational roles, and premissory and conclusory roles of sentences. Implicational role inclusions. General constructive correlation of operations on implicational roles and connective definitions in sequent calculi generate semantic connective rules that are sound and complete for NMMS.

We have demonstrated how to construct an isomorphism between (1) and (2), a general constructive correlation between (3) and (4) up to soundness and completeness, and how to use both (3) and (4) to capture what is common to (1) and (2).
Proposal: Define *reason relations* functionally, from above, as what can be specified in all four of these kinds of rational metavocabulary, so that they stand in these relations.

Material is from Chapter 5 of Ulf Hlobil and Robert Brandom *Reasons for Logic, Logic for Reasons: Pragmatics, Semantics, and Conceptual Roles* [Routledge, 2024].