Three Approaches to Finitude in Belief Change

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Abstract.
Since neither a human mind nor a computer can deal directly with infinite structures, well-behaved models of belief change should operate exclusively on belief states that have a finite representation. Three ways to achieve this without resorting to a finite language are investigated: belief bases, specified meet contraction, and focused propositional extenders. Close connections are shown to hold between the three approaches.

Keywords: finitude, finiteness, belief change, belief base, specified meet contraction, sentential selector, propositional extender, Grove spheres

1. Introduction

Neither a human mind nor a computer can deal directly with infinite entities. Therefore, a realistic representation of belief change should reflect the finiteness of actual belief systems. In the present contribution, I will introduce and investigate three approaches to belief contraction that satisfy this requirement. One of these approaches is based on a suggestion by Wlodek Rabinowicz.

After formal preliminaries have been provided in Section 2, the requirements of finiteness are developed in Section 3. The three models will be introduced in Section 4, and their interrelations are investigated in Section 5.

2. Formal preliminaries

The belief-representing sentences form a language $\mathcal{L}$. Sentences, i.e. elements of this language, are represented by lowercase letters $(p, q, \ldots)$ and sets of sentences by capital letters. The language contains the usual truth-functional connectives: negation ($\neg$), conjunction (&), disjunction ($\vee$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$).

To express the logic, a Tarskian consequence operator Cn will be used. Intuitively speaking, for any set A of sentences, $\mathrm{Cn}(A)$ is the set of logical consequences of A. Cn is a function from sets of sentences to sets of sentences. It satisfies the standard conditions: inclusion ($A \subseteq \mathrm{Cn}(A)$), monotony (If $A \subseteq B$, then $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(B)$) and iteration

Printed from: *Hommage à Wlodek. Philosophical Papers Dedicated to Wlodek Rabinowicz.* Ed. T. Rønnow-Rasmussen, B. Petersson, J. Josefsson & D. Egonsson, 2007. www.fil.lu.se/hommageawlodek

$(\mathrm{Cn}(A) = \mathrm{Cn}(\mathrm{Cn}(A)))$ Furthermore, Cn is assumed to be supraclassical (if p follows from A by classical truth-functional logic, then $p \in \mathrm{Cn}(A)$), and to satisfy the deduction property ($q \in \mathrm{Cn}(A \cup \{p\})$ if and only if $(p \rightarrow q) \in \mathrm{Cn}(A)$).

A is a belief set if and only if $A = \mathrm{Cn}(A)$. A set A is *finite-based* if and only if there is some finite set A' such that $\mathrm{Cn}(A) = \mathrm{Cn}(A')$.

For any finite-based set A, $\&A$ (the "conjunction of A") is a sentence such that $\mathrm{Cn}(A) = \mathrm{Cn}(\{\&A\})$. Furthermore, for any set A of sentences, $C_{\&}(A)$ (the "conjunctive closure of A") is the set $\{\&(A') \mid \varnothing \neq A' \subseteq A\}$.

K denotes a belief set. $\mathrm{Cn}(\varnothing)$ is the set of tautologies. $X \vdash p$ is an alternative notation for $p \in \mathrm{Cn}(X)$ and $\vdash p$ for $p \in \mathrm{Cn}(\varnothing)$.

For any sets A and X, the remainder set $A \perp X$ ("A remainder X") is the set of inclusion-maximal subsets of A that do not imply any element of X. In other words, a set B is an element of $A \perp X$ if and only if B is a subset of A that does not imply any element of X, and there is no set B' not implying any element of X such that $B \subset B' \subseteq A$. For any sentence p, we define $A \perp p = A \perp \{p\}$. $\mathbb{W} = \mathcal{L} \perp (p\&\neg p)$ is the set of maximal consistent subsets of the language.

In one of the proofs we will make use of the *upper bound property*, namely: If $X \subseteq A$ and X implies no element of B, then there is some X' such that $X \subseteq X' \in A \perp B$. As was observed by Alchourrón and Makinson (1981, p. 129), the upper bound property follows from compactness and Zorn's lemma.

Expansion, denoted $+$, is the operation such that $K + p = \mathrm{Cn}(K \cup \{p\})$ *Full meet contraction*, denoted $\sim$, is the operation such that if $K \perp X$ is non-empty, then $K \sim X = \cap(K \perp X)$ and otherwise $K \sim X = K$. Furthermore, $K \sim p = K \sim \{p\}$.

3. The requirements of finitude

There are several ways to operationalize the rejection of infinite objects. Perhaps the most obvious is:

$\mathcal{L}$ is finite (*finite language*)

The same effect can be obtained if the language is infinite but the logic that is applied to it does not distinguish between more than a finite number of equivalence classes of sentences in the language:

If $X \subseteq \mathcal{L}$ and $\mathrm{Cn}(\{p\}) \neq \mathrm{Cn}(\{q\})$ for all $p, q \in X$, then X is finite (*finite logic*)

Clearly, the logic is finite if the language is finite. In a 1988 paper, two members of the AGM trio, Peter Gärdenfors and David Makinson, expressed support of *finite logic*. They wrote:

> "In all applications, the knowledge sets [belief sets] will be finite in the sense that the consequence relation $\vdash$ partitions the elements of K into a finite number of equivalence classes.' (Gärdenfors and Makinson 1988)

However, this is unnecessarily harsh. An infinite language can be built from a finite set of components, and finite epistemic agents can make use of unlimited linguistic constructs such as the series of natural numbers. For a simple example of this, for every positive integer n, let π_n denote that the Roman Catholic Church has at present at least one and at most n popes. I believe in every sentence in the infinite sequence $\pi_1, \pi_2, \pi_3, ...$ for the simple reason that I believe in π_1 that implies all the rest. Holding this set of beliefs is not an infinitistic feat, but it nevertheless violates *finite logic* since no two sentences in the sequence $\pi_1, \pi_2, \pi_3, ...$ are logically equivalent. The finiteness of actual belief systems seems to be weaker than what is expressed by this postulate.

Weaker and more plausible requirements of finiteness can be made. First:

> K is finite-based (*finite-based origin*)

Secondly, it can be required that the belief sets that are obtained through contraction also have finite representations. This can be expressed as follows:

> For all p, $K \div p$ is finite-based (*finite-based outcomes*) (Hansson 1993)

Neither *finite-based origin* nor *finite-based outcomes* is in conflict with the popes example. For the infinite sequence $\pi_1, \pi_2, \pi_3, ...$ to be included in K or $K \div p$ it is sufficient that π_1 is included in the finite representation (base) of K (respectively $K \div p$). It should also be noted that *finite-based origin* follows from *finite-based outcomes* if there is some p (such as a tautology or a non-element of K) such that $K \div p = K$.

Thirdly and finally, it can be required that although there may be infinitely many sentences by which the belief set can be contracted, only a finite number of belief sets can be obtained through contraction:

> $\{K' \mid (\exists p)(K' = K \div p)\}$ is finite (*finite range*) (Hansson 1993)

This postulate is also compatible with the popes example. With reasonable background beliefs about the Roman Catholic Church, π_2, π_3 , etc. are believed only as a consequence of belief in π_1. They all stand or fall with π_1, so that if one of them is lost, then the rest of them will be lost as well. Thus, e.g., $K \div \pi_8 = K \div \pi_9$. (There may be ways to make me believe in π_9 but not in π_8, but this will have to involve the acquisition of new beliefs, and cannot therefore be expressed by an operation of contraction.)

Finite range may appear to be a less indispensable requirement in a finitistic system than *finite-based outcomes.* The primary requirement is that each belief state should have a finite structure, not that the number of potential such states should be finite. It is instructive to compare to revision. We should expect *finite-based outcomes* to hold for revisions of a finite-based belief set, in other words $K * p$ should be finite-based. On the other hand, *finite range* is not a plausible requirement on revision, in other words $\{K' \mid (\exists p)(K' = K * p)\}$ may well be infinite. However, there is an important difference between contraction and revision that should be noted at this point: The operation of contraction should add nothing new to the belief set. Since K does not contain resources for making infinitely many distinctions, it should not have the inherent infinite structure (namely divisibility in infinitely many different ways) that is required for *finite range* to be satisfied.

Finite range and *finite-based outcomes* can be combined into the following property:

> There is a finite set A such that for every sentence p, $K \div p = \mathrm{Cn}(A')$ for some $A' \subseteq A$. (*finitude*)

OBSERVATION 1. *A contraction operator $\div$ for K satisfies* finitude *if and only if it satisfies both* finite-based outcomes *and* finite range.

PROOF OF OBSERVATION 1: For one direction, suppose that *finitude* is satisfied. Then there is a finite set A such that for every p, $K \div p = \mathrm{Cn}(A')$ for some $A' \subseteq A$. Since every subset of A is finite, *finite-based outcomes* holds. Since A only has a finite number of subsets, *finite range* holds as well.

For the other direction, suppose that *finite-based outcomes* and *finite range* both hold. Let $K_1, ...K_n$ be all the possible outcomes of contractions. For each K_k, with $1 \leq k \leq n$, there is a finite set A_k such that $K_k = \mathrm{Cn}(A_k)$. Let $A = A_1 \cup ... \cup A_n$. Then A is the finite set that is needed for *finitude* to hold.

As should be clear from the above, I consider *finitude* to be a reasonable general requirement on rational belief change. Nevertheless, for reasons of clarity, its two components will be treated as separate requirements in what follows.

Finite-based outcomes and *finite range* are both conditions on the range of possible contraction outcomes, and they are both neutral with respect to the assignment of these outcomes to specific input sentences. They are therefore range conditions, in the following sense:

DEFINITION 1. *A condition $\mathcal{C}$ on operators of contraction is a* range condition *if and only if it holds for all contraction operators $\div$ and $\div'$ that if $\{K \div p \mid p \in \mathcal{L}\} = \{K \div' p \mid p \in \mathcal{L}\}$, then $\mathcal{C}$ holds for $\div$ if and only if it holds for $\div'$.*

Closure ($K \div p = \text{Cn}(K \div p)$) is another range condition. In addition to being range conditions, *closure* and *finite-based outcomes* have in common that they both require that the contraction outcome has the same type of belief representation as the initial belief set. A contraction of a finite-based, logically closed set should result in a new finite-based, logically closed set, not in a belief set that has no finite base or in a set that is not logically closed.

Since this study is devoted to finiteness properties, the focus will be on range conditions. Clearly, in a more general study of contraction, other conditions will have a more prominent place. However, the following two conditions should be mentioned, since they summarize the most elementary additional conditions that a contraction operator is conventionally required to satisfy:

If $p \notin \text{Cn}(\varnothing)$ then $p \notin \text{Cn}(K \div p)$. (*success*)

If $p \notin K \setminus (K \div q)$ for all q, then $K \div p = K$. (*futility*)

Futility says essentially that if p cannot be removed from K by $\div$, then contraction by p leaves K unchanged. As the following observation shows, *futility* summarizes two well-known postulates, namely *vacuity* (If $p \notin \text{Cn}(K)$, then $K \div p = K$) (Alchourrón et al 1985) and *failure* (If $p \in \text{Cn}(\varnothing)$, then $K \div p = K$.) (Fuhrmann and Hansson 1994).

OBSERVATION 2. *Let K be a belief set and $\div$ an operation on K that satisfies* closure *and* success*. Then $\div$ satisfies* futility *if and only if it satisfies both* vacuity *and* failure*.*

PROOF OF OBSERVATION 2: For one direction, let *vacuity* and *failure* hold, and let $p \notin K \setminus (K \div q)$ for all q. If $p \notin K$, then it follows from *vacuity* that $K \div p = K$. In the principal case when $p \in K$, we have $p \in K \div q$ for all q, thus $p \in K \div p$, thus $p \in \text{Cn}(K \div p)$. It follows from *success* that $p \in \text{Cn}(\varnothing)$ and from *failure* that $K \div p = K$.

For the other direction, let *futility* be satisfied. To show that *vacuity* holds, let $p \notin \mathrm{Cn}(K)$. Then $p \notin K \setminus (K \div q)$ for all q, and *futility* yields $K \div p = K$. To show that *failure* holds, let $p \in \mathrm{Cn}(\varnothing)$. Then *closure* yields $p \in K \div q$ for all q, thus $p \notin K \setminus (K \div q)$ for all q, and *futility* yields $K \div p = K$.

Success expresses the assumption that all non-tautological sentences are contractible. Arguably, this is not a fully realistic condition. Alternatively, we may assume that some non-tautologies are so entrenched in the belief system that they cannot be contracted, at least not in a single step. (Fermé and Hansson 2001) In such a framework, the following weakened variant of the success condition is more adequate:

> If $p \notin \mathrm{Cn}(K \div q)$ then $p \notin \mathrm{Cn}(K \div p)$. (*persistence*) (Fermé and Hansson 2001, p 86)

4. Three approaches

Just as in the standard AGM framework we will be concerned with the contraction of a belief set K by a sentence p. Thus, neither multiple nor iterated contraction will be covered. Since the subject of this study is finitistic belief contraction, it will be assumed that K is finite-based. As I have shown elsewhere (Hansson 2006a), the standard AGM operator of partial meet contraction does not preserve finite-basedness. In other words, partial meet contraction of a finite-based belief set by a single sentence can result in a contraction outcome that is not finite-based. In what follows, three alternative approaches will be proposed, in which finite-basedness is preserved.

4.1. Belief bases

Probably the most obvious way to make belief contraction finitistic, and the only way with some tradition in the area, is the use of belief bases. A belief base is a (typically finite) subset of the belief set K that has the same logical closure as K.

DEFINITION 2. *(1) Any set B of sentences is a belief base.*

(2) Let K be a belief set (theory). Then a set B of sentences is a belief base for K if and only if $K = \mathrm{Cn}(B)$.

In base-generated contraction (Fuhrmann 1991, Hansson 1993), there is a belief base B for K, such that for each sentence p, $K \div p = \mathrm{Cn}(B')$ for some subset B' of B. More generally, base contraction can be defined as follows:

DEFINITION 3. *(1) A base function for the belief set K is a function b from sentences to sets of sentences, such that for all p, $b(p) \subseteq K$.*

(2) An operation $\div$ for K is base-generated if and only if there is some base function b such that for all p, $K \div p = \mathrm{Cn}(b(p))$.

As should be obvious, the belief base can be reconstructed from b as $\cup\{b(p) \mid p \in \mathcal{L}\}$. The following observation provides an axiomatic characterization of this very general construction, and identifies the conditions under which it satisfies the finiteness properties.

OBSERVATION 3. *(1) An operator $\div$ on the belief set K is a base-generated contraction if and only if $\div$ satisfies* closure *and* inclusion*.*

Furthermore, this $\div$

(2) satisfies finite-based outcomes *if and only if it is generated from a base function b such that for all p, $b(p)$ is finite-based.*

(3) satisfies finite range *if and only if it is generated from a base function b such that $\{\mathrm{Cn}(b(p)) \mid p \in \mathcal{L}\}$ is finite.*

PROOF OF OBSERVATION 3: Part 1: For one direction, let $\div$ be a base-generated contraction. It follows from Definition 3 that *closure* and *inclusion* are satisfied. For the other direction, let $\div$ satisfy *closure* and *inclusion.* For each p, let $b(p) = K \div p$.

Parts 2 and 3 follow directly from Definition 3.

For any belief base B, $\{\mathrm{Cn}(X) \mid X \subseteq B\}$ is the set of possible contraction outcomes that can be generated from B. We can call this its range, and define two belief bases as range equivalent if they have the same range.

DEFINITION 4. *Two belief bases B and B' are* range equivalent *if and only if $\{\mathrm{Cn}(X) \mid X \subseteq B\} = \{\mathrm{Cn}(X) \mid X \subseteq B'\}$.*

OBSERVATION 4. *Two belief bases B and B' are range equivalent if and only if each element of $C_{\&}(B)$ is logically equivalent with some element of $C_{\&}(B')$, and vice versa.*

PROOF OF OBSERVATION 4: For one direction, let B and B' be range equivalent and let $p \in C_{\&}(B)$. Then there is some $X \subseteq B$ such that $p = \&X$. Since B and B' are range equivalent there is some $X' \subseteq B'$ such that $\mathrm{Cn}(X') = \mathrm{Cn}(X)$. Then $\&X' \in C_{\&}(B')$.

For the other direction, let the condition given in the observation be satisfied, and let Z belong to the range of B, i.e. $Z = \mathrm{Cn}(X)$ for some $X \subseteq B$. It follows from the condition that there is some $X' \subseteq B'$ such that $\vdash \&X \leftrightarrow \&X'$, thus we have $Z = \mathrm{Cn}(X) = \mathrm{Cn}(\{\&X\}) =$

$\mathrm{Cn}(\{\&X'\}) = \mathrm{Cn}(X')$. This is sufficient to show that range equivalence holds.

4.2. Specified meet contraction

The recently proposed operation of specified meet contraction (Hansson 2006a, 2006b) has turned out to be a convenient method to construct finitistic contraction operators. Specified meet contraction makes use of full meet contraction ($\sim$) as a building-block in reconstructing other operations of contraction, combining it with a sentential selector. This is a function that, intuitively speaking, takes us from the given contraction input p to some other sentence $f(p)$ that represents the part of the input that is "really" going to be completely removed from the belief set. (Hence, if in the contraction by $p\&q$, we remove p but let q remain, then $f(p\&q)$ should not imply q.) The formal definition is as follows:

DEFINITION 5. (Hansson 2006a) *(1) A* sentential selector *is a function from and to $\mathcal{L}$.*

(2) An operation $\div$ on K is an operation of specified meet contraction *if and only if there is a sentential selector f such that for all p, $K \div p = K \sim f(p)$.*

The reason why specified meet contraction is finitistic is that full meet contraction, in sharp contrast to partial meet contraction, always yields a finite-based outcome if the original belief set is finite-based:

OBSERVATION 5. (Hansson 2006a) *If K is finite-based, then so is $K \sim p$ for all p.*

LEMMA 1. *Let K be a logically closed set and $\sim$ the operator of full meet contraction for K. Then for all sentences $p \in K$:*
$K \sim p = K \cap \mathrm{Cn}(\{\neg p\})$.

PROOF OF LEMMA 1: See Alchourrón and Makinson 1982, pp. 18-19 or Hansson 1999, pp. 125-126.

PROOF OF OBSERVATION 5: Due to Lemma 1, $K \sim p = K \cap \mathrm{Cn}(\{\neg p\})$, and since K is finite-based we then have $K \sim p = \mathrm{Cn}(\{\&K \vee \neg p\}) = \mathrm{Cn}(\{p \rightarrow \&K\})$.

A surprisingly wide range of contraction operators can be constructed in this way:

OBSERVATION 6. (Hansson 2006b) *An operation* $\div$ *on a finite-based belief set* K *is a specified meet contraction if and only if it satisfies:*

$K \div p \subseteq K$ (inclusion)

$K \div p = \mathrm{Cn}(K \div p)$ (closure)

$K \div p$ *is finite-based* (finite-based outcomes)

PROOF OF OBSERVATION 6: For one direction, let $K \div p = K \sim f(p)$. *Inclusion* and *closure* follow from the properties of full meet contraction. It follows from Observation 5 that *finite-based outcomes* is satisfied.

For the other direction, let $\div$ satisfy the three postulates. Due to *finite-based outcomes*, $\&(K \div p)$ is well-defined. We can therefore define $f(p)$ so that:

If $K \div p \subset K$, then: $f(p) = \&(K \div p) \rightarrow \&K$.

Otherwise: $f(p) \notin K \setminus \mathrm{Cn}(\varnothing)$.

In order to verify the construction we need to show that the identity $K \div p = K \sim f(p)$ holds. Due to *inclusion*, either $K \div p = K$ or $K \div p \subset K$. In the former case, it follows directly from our definition of $f(p)$ and the properties of full meet contraction that $K \sim f(p) = K$. In the latter case we have:

$K \div p = \mathrm{Cn}(\{\&(K \div p)\})$ (*closure*, *finite-based outcomes*)

$= \mathrm{Cn}(\{\&K \vee \neg(\&(K \div p) \rightarrow \&K)\})$

(Since $\vdash \&K \rightarrow \&(K \div p)$ that follows from *inclusion*)

$= \mathrm{Cn}(\{\&K\}) \cap \mathrm{Cn}(\{\neg(\&(K \div p) \rightarrow \&K)\})$

$= K \cap \mathrm{Cn}(\{\neg(\&(K \div p) \rightarrow \&K)\})$

$= K \sim (\&(K \div p) \rightarrow \&K)$ (Lemma 1)

$= K \sim f(p)$.

It can be concluded from Observation 6 that specified meet contraction is sufficiently general to cover all plausible contraction operations.

The following observation introduces the other finiteness property:

OBSERVATION 7. *An operator* $\div$ *of specified meet contraction, with the sentential selector* f*, satisfies* finite range *if and only if* $\{f(p) \mid p \in \mathcal{L}\} \cap K$ *has only a finite number of logically non-equivalent elements.*

LEMMA 2. *Let* $p, q \in K$*. Then* $K \sim p = K \sim q$ *if and only if* $\vdash p \leftrightarrow q$.

PROOF OF LEMMA 2: For one direction, let $K \sim p = K \sim q$. It follows from Lemma 1 that $\vdash (p \rightarrow \&K) \leftrightarrow (q \rightarrow \&K)$. From this, $\vdash \&K \rightarrow p$, and $\vdash \&K \rightarrow q$ it follows that $\vdash p \leftrightarrow q$.

For the other direction, let $\vdash p \leftrightarrow q$. Then $K \sim p = K \sim q$ follows directly from the properties of full meet contraction.

PROOF OF OBSERVATION 7: For one direction, let $\div$ satisfy *finite range* and suppose for contradiction that $\{f(p) \mid p \in \mathcal{L}\} \cap K$ has an infinite number of logically non-equivalent elements. It follows from Lemma 2 that $\{K \sim f(p) \mid f(p) \in K\}$ is infinite as well, contrary to *finite range*.

For the other direction, let $\{f(p) \mid p \in \mathcal{L}\} \cap K$ have a finite number of non-equivalent elements. Then, according to Lemma 2, $\{K \sim f(p) \mid f(p) \in K\}$ is finite. It follows from the properties of full meet contraction that $\{K \sim f(p) \mid f(p) \notin K\} = \{K\}$, Thus, $\{K \div p \mid p \in \mathcal{L}\} = \{K \sim f(p) \mid f(p) \in K\} \cup \{K \sim f(p) \mid f(p) \notin K\}$ is finite, i.e. *finite range* is satisfied.

Based on Lemma 2, an identity criterion for specified meet contraction is readily obtainable.

OBSERVATION 8. *Let $\div$ and $\div'$ be the operators of specified meet contraction for a belief set K that are based on the sentential selectors f and f', respectively. Then $\div$ and $\div'$ are identical (i.e., $K \div p = K \div' p$ for all p) if and only if for all p:*

(1) $f(p) \in K \setminus \mathrm{Cn}(\varnothing)$ if and only if $f'(p) \in K \setminus \mathrm{Cn}(\varnothing)$, and

(2) if $f(p), f'(p) \in K \setminus \mathrm{Cn}(\varnothing)$ then $\vdash f(p) \leftrightarrow f'(p)$.

PROOF OF OBSERVATION 8: $K \div p = K \div' p$ holds for all p if and only if (i) for all p, $K \div p = K$ iff $K \div' p = K$, and (ii) whenever $K \div p \subset K$, then $K \div p = K \div' p$. It follows from the properties of full meet contraction that (i) holds if and only if (1) is satisfied. It follows from Lemma 2 that (ii) holds if and only if (2) is satisfied.

4.3. A geometric model

When I showed some preliminary results on specified meet contraction to Wlodek Rabinowicz, emphasizing its finitistic properties, he urged me to clarify what this would mean in a propositional framework in which geometric intuitions can be applied.

In such a framework, sets of possible worlds are used as an alternative representation of belief states. A belief state can be represented by the proposition (set of possible worlds) that contains exactly those possible worlds that are compatible with the agent's beliefs. (Grove 1988, Lindström and Rabinowicz 1991) There is a close connection between propositions and belief sets. The belief set K and the proposition (set of possible worlds) $\mathcal{W}$ represent the same belief state if and only if it holds for each possible world W that:

$W \in \mathcal{W}$ if and only $K \subseteq W$. (For details, see Hansson 1999, p. 221)

It follows that if the belief set K and the proposition $\mathcal{W}$ represent the same belief state, then $\mathcal{W}$ consists of exactly those possible worlds that contain K. This set of possible worlds will be denoted by $[K]$. Hence, a proposition $\mathcal{W}$ represents the same belief state as a belief set K if and only if $\mathcal{W} = [K]$. For every belief set K there is a set $[K]$ of possible worlds (a proposition) that represents the same belief state as K. If K is the inconsistent belief set, then $[K] = \varnothing$. Otherwise, $[K]$ is a non-empty set of possible worlds.

For any sentence p, $[p]$ is an abbreviation of $[\mathrm{Cn}(\{p\})]$. A proposition $\mathcal{W}$ is *sentence-representing* if and only if there is some sentence p such that $\mathcal{W} = [p]$. If the logic is infinite (has an infinite number of equivalence classes), then the set of propositions has higher cardinality than the set of sentences, and it follows that not all propositions are sentence-representing.

Possible world models can be used for contraction. In contraction, a restriction on what worlds are possible (compatible with the agent's beliefs) is removed. Thus, the set of possibilities is enlarged. Therefore the contraction of $[K]$ by $[p]$ is a superset of $[K]$. The following is a fully general selection mechanism for such contractions:

DEFINITION 6. *(1) A* propositional extender *for K is a function g from and to the set of propositions such that for all $\mathcal{W}$, $[K] \subseteq g(\mathcal{W})$.*

(2) The propositional extender g is focused *if and only if it holds for all sentences p that $g([p])$ is sentence-representing.*

(3) An operator $\div$ on K is a propositional contraction *if and only if there is a propositional extender g such that for all p, $K \div p = \cap g([p])$.*

(Alternatively, propositional contraction can be defined as an operator $\ominus$ on the proposition $[K]$ such that $[K] \ominus p = g([p])$.) Propositional contraction can be axiomatically characterized as follows:

OBSERVATION 9. *An operator $\div$ on a finite-based belief set K is a propositional contraction if and only if it satisfies:*

$K \div p \subseteq K$ (inclusion) *and*

$K \div p = \mathrm{Cn}(K \div p)$ (closure).

PROOF OF OBSERVATION 9: For one direction, let $\div$ be a propositional contraction. It follows directly from Definition 6 that *inclusion* and *closure* are satisfied. For the other direction, let $\div$ satisfy *inclusion* and *closure*. Define g such that for all p:

$g([p]) = \{W \in \mathbb{W} \mid K \div p \subseteq W\}$

It follows from *inclusion* that $[K] \subseteq g([p])$ for all p, so that g satisfies the criterion of Definition 6 for being a propositional extender. We then have:

$\cap g([p]) = \cap\{W \in \mathbb{W} \mid K \div p \subseteq W\}$
$= \mathrm{Cn}(K \div p)$
$= K \div p$ (*closure*)

OBSERVATION 10. *Let* $\div$ *be the propositional contraction on a finite-based belief set* K *that is based on the propositional extender* g*. Then* $\div$ *satisfies* finite-based outcomes *if and only if* g *is focused.*

LEMMA 3. *Let* A *and* B *be logically closed sets. Then:*
(1) $[A] \subseteq [B]$ *if and only if* $B \subseteq A$*, and*
(2) $[A] = [B]$ *if and only if* $A = B$*.*

PROOF OF LEMMA 3: See Hansson 1999, pp. 52-53.

PROOF OF OBSERVATION 10: For one direction, let g be focused. Then for each p there is some q such that $g([p]) = [q]$, and we have $K \div p = \cap(g([p])) = \cap([q]) = \mathrm{Cn}(\{q\})$ so that $K \div p$ is finite-based.

For the other direction, let $\div$ satisfy *finite-based outcomes*. We have $K \div p = \cap(g([p]))$. Since $K \div p$ is finite-based it follows that $\cap[\&(K \div p)] = \cap g([p])$. Lemma 3 yields $g([p]) = [\&(K \div p)]$. Since this holds for all p, g is focused.

OBSERVATION 11. *Let* $\div$ *be the propositional contraction on a finite-based belief set* K *that is based on the propositional extender* g*. Then* $\div$ *satisfies* finite range *if and only if* $\{g(p) \mid p \in \mathcal{L}\}$ *is finite.*

PROOF OF OBSERVATION 11: Obvious.

DEFINITION 7. *Two propositional extenders* g *and* g' *are* contraction equivalent *if and only if, for all* p, $K \div p = K \div' p$*, where* $\div$ *and* $\div'$ *are the propositional contractions generated by* g *and* g'*, respectively.*

OBSERVATION 12. *Two propositional extenders* g *and* g' *are* contraction equivalent *if and only if they are identical (i.e., if and only if* $g(p) = g'(p)$ *for all* p*).*

PROOF OF OBSERVATION 12: $K \div p = K \div' p$ holds if and only if $\cap g([p]) = \cap g'([p])$, thus according to Lemma 3 if and only if $g([p]) = g'([p])$.

5. Connections between the three approaches

The major results on range conditions from the previous section are summarized in the table.

	Base-generated contraction	Specified meet contraction	Propositional contraction
Closure	Always	Always	Always
Inclusion	Always	Always	Always
Finite-based outcomes	$b(p)$ is finite-based for all p	Always	g is focused
Finite range	$\{\text{Cn}(b(p)) \mid p \in \mathcal{L}\}$ is finite	$\{f(p) \mid p \in \mathcal{L}\} \cap K$ is logically finite	$\{g(p) \mid p \in \mathcal{L}\}$ is finite

Finally, the following two observations connect the three approaches:

OBSERVATION 13. *Let K be a finite-based belief set, g a focused propositional extender and b a base function for K such that for all p, $b(p)$ is finite-based. Furthermore, let $\div_g$ be the contraction operator that is based on g and $\div_b$ the contraction operator that is based on b. Then the following two conditions are equivalent:*
(1) $K \div_b p = K \div_g p$ for all p.
(2) $g([p]) = [b(p)]$.

PROOF OF OBSERVATION 13:
$K \div_b p = K \div_g p$
if and only if $\text{Cn}(b(p)) = \cap g([p])$ (Definitions 3 and 6)
if and only if $[b(p)] = g([p])$ (Lemma 3)

OBSERVATION 14. *Let K be a finite-based belief set, f a sentential selector and b a base function for K such that for all p, $b(p)$ is finite-based. Furthermore, let $\div_f$ be the contraction operator that is based on f and $\div_b$ the contraction operator that is based on b. Then the following three conditions are equivalent:*
(1) $K \div_b p = K \div_f p$ for all p.
(2) $\vdash \&b(p) \leftrightarrow (f(p) \rightarrow \&K)$ for all p.
(3) $\vdash f(p) \leftrightarrow (\&b(p) \rightarrow \&K)$ for all p.

PROOF OF OBSERVATION 14: To see that (1) and (2) are equivalent, note that $K \div_b p = \mathrm{Cn}(b(p))$ according to Definition 3 and $K \div_f p = \mathrm{Cn}(\{f(p) \to \&K\})$ according to Definition 5 and Lemma 1.

To see that (2) and (3) are equivalent it is sufficient to note that since $\vdash K \to f(p)$ and $\vdash K \to \&b(p)$, $\&b(p) \leftrightarrow (f(p) \to \&K)$ and $f(p) \leftrightarrow (\&b(p) \to \&K)$ are equivalent.

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